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The ideal Bose gas in non-integer dimensions and liquid ^{4}He in porous media

Sang-Hoon Kim†, Chul Koo Kim‡ and Kyun Nahm§

† Division of Liberal Arts, Mokpo National Maritime University, Mokpo 530-729, Korea
‡ Department of Physics and Institute for Mathematical Sciences, Yonsei University, Seoul 120-749, Korea
§ Department of Physics, Yonsei University, Wonju 220-710, Korea

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Abstract. Physical properties of the ideal Bose gas with non-integer dimensions between $D = 2$ and $D = 3$ are theoretically investigated. Calculation shows that there exists a hierarchy of condensation transitions with changing fractional dimensionality. The results obtained on the specific heat and the condensed density in non-integer dimensions are similar to those for liquid helium-4 in porous media. This results suggest that the geometrical factor may be important for the physical properties of liquid helium-4 in porous media.

1. Introduction

Recently, liquid helium-4 in highly connected porous structures (glass plate, xerogel, aerogel, graphite, fine powders, steel, German silver, plastic films, etc) has been studied intensively [1–6]. However, so far no satisfactory explanation of the experimental observations on these materials has been achieved [7–13]. Since porous media can be treated as solids with non-integer dimensions [11–16], we believe that it is imperative to study the extent of the dimensionality contribution to the physical properties of liquid helium-4 in order to understand the experimental results.

In this paper, we examine the physical properties of the ideal Bose gas with non-integer dimensions between $D = 2$ (the thin-film limit) and $D = 3$ (the bulk limit). The results are compared with experimental results obtained for liquid helium-4 in porous media. Surprisingly, we find that most of the salient features of experimental observations on liquid helium-4 in porous solids can be found in the theoretical results for the ideal-gas model in non-integer dimensions. This suggests that the dimensionality contribution is a dominant factor in determining the physical properties of liquid helium-4 in porous media.

2. The ideal Bose gas in non-integer dimensions

The ideal-Bose-gas system at integer dimensions was first studied long ago [17, 18]. However, study at non-integer dimensions was first carried out only quite recently [16, 19]. The density of states which is essential for calculating thermodynamic properties of the Bose gas in D dimensions, where D is any real number, is given by [14]

$$\rho_D(E) = a_D E^{D/2-1} \tag{1}$$

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where a_D is a D-dimensional coefficient which is known to be

$$a_D = \frac{V(D)}{\Gamma(D/2)}\left(\frac{m}{2\pi\hbar^2}\right)^{D/2}. \tag{2}$$

Here, Γ is the Gamma function, m the mass, and $V(D)$ the D-dimensional volume.

In order to obtain physical quantities for the D-dimensional Bose gas, it is necessary to obtain the grand partition function in D dimensions:

$$\ln Q(z, v, t) = -\ln(1-z) - \int_0^\infty \ln(1 - z\mathrm{e}^{-\beta E})\rho_D(E)\,\mathrm{d}E = -\ln(1-z) + \frac{V}{\lambda^D} g_{D/2+1}(z) \tag{3}$$

where z is the thermodynamic fugacity defined by $z = \mathrm{e}^{\beta\mu}$, $E(\boldsymbol{p}) = p^2/2m$, and $\lambda(T)$ is the thermal wavelength defined by $\sqrt{2\pi\hbar^2/mk_BT}$. With the coefficient a_D in equation (2), we can readily calculate the grand partition function in D dimensions. $g_s(z)$ is the Bose-gas function defined by

$$g_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}. \tag{4}$$

The coefficients s and z are restricted to the $s > 0$ and $0 \leqslant z \leqslant 1$ regions. The Bose-gas function can be also extended to non-integer dimensions [20], and has an integral expression:

$$g_s(z) = -\frac{1}{\Gamma(s-1)} \int_0^\infty \mathrm{d}x\, x^{s-2} \ln(1 - z\mathrm{e}^{-x}). \tag{5}$$

Note that this representation is valid only when $s > 1$.

Using the above expressions, we obtain the average number of particles in D dimensions:

$$N = z\frac{\partial}{\partial z} \ln Q(z, v, t) = \sum_{\boldsymbol{p}\neq\boldsymbol{0}} \langle n_{\boldsymbol{p}}\rangle + \langle n_{\boldsymbol{0}}\rangle = \frac{V}{\lambda^D} g_{D/2}(z) + \frac{z}{1-z}. \tag{6}$$

We use the above equations to calculate thermodynamic properties in non-integer dimensions in the following.

3. Physical properties of the ideal Bose gas in non-integer dimensions and liquid helium-4 in porous media

The specific heat of the ideal Bose gas in non-integer-dimensional space can readily obtained from the grand partition function Q in equation (3) [17, 18]:

(a) When $T \leqslant T_c$,

$$\frac{C_V(T)}{Nk_B} = \frac{D}{2}\left(\frac{D}{2}+1\right)\frac{v}{\lambda^D} g_{D/2+1}(1). \tag{7}$$

(b) When $T > T_c$,

$$\frac{C_V(T)}{Nk_B} = \frac{D}{2}\left(\frac{D}{2}+1\right)\frac{v}{\lambda^D} g_{D/2+1}(z) - \left(\frac{D}{2}\right)^2 \frac{g_{D/2}(z)}{g_{D/2-1}(z)}. \tag{8}$$

The $C_V(T)$ curves for several values of non-integer dimensions are plotted as functions of temperature in figure 1. We observe that the cusp disappears when the dimension is less than 3, and the peak height becomes smaller with decreasing dimensionality. Also, figure 1 shows that there is a systematic crossover in low-temperature regions.

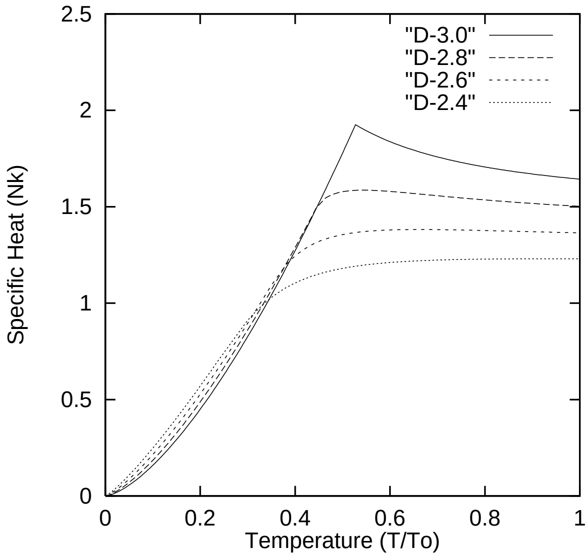

Figure 1. Specific heat functions of the ideal-Bose-gas system between two and three dimensions. k is k_B, and the unit of temperature is T_0 where $T_0 \simeq 5.42$ K $(=g_{D/2}(1)^{2/D}T_c(D))$.

Since the present model completely neglects the mutual interactions between the Bose particles, it cannot be directly applied to liquid helium-4. However, it is interesting to study whether the above dimensionality-dependent characteristics appear also in the specific heat data for liquid helium-4 in porous media. Figure 2 summarizes the specific heat data for liquid helium-4 in porous media [1–3]. Careful examination of figure 2 shows that the specific heat data for liquid helium-4 in porous media have the following properties:

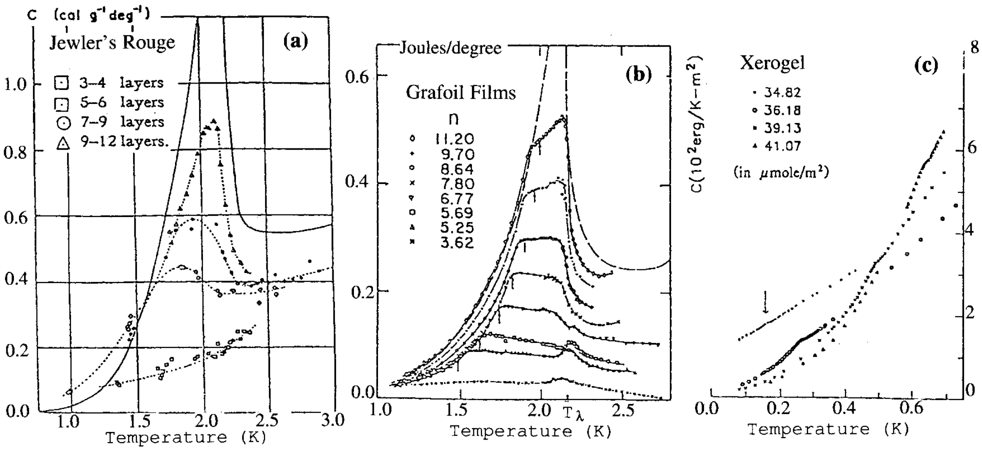

Figure 2. Heat capacity measurements for helium-4 in various porous media. (a) Jeweller's rouge (powder) [1]. (b) Grafoil [2]. (c) Xerogel [3]. Some data have been deleted for clarity. Arrows indicate temperatures below which the film relaxation times go to zero.

(a) The bulk-like sharp cusp disappears and the peak gets smaller and rounder with decreasing dimensionality.
(b) There exists a systemic crossover at low temperatures.

We note that these characteristics are the same as the ones that we found in figure 1 for the ideal Bose gas in non-integer dimensions. This qualitative similarity between the two systems suggests that the dimensionality has a strong influence on the physical properties of liquid helium-4 in porous media.

The critical temperature, T_c, for condensation of the ideal Bose gas in non-integer dimensions can be obtained, too, from equation (6):

$$k_B T_c = \frac{2\pi\hbar^2}{m}\frac{1}{[v g_{D/2}(1)]^{2/D}}. \tag{9}$$

The almost linear behaviour of T_c as a function of dimensionality is plotted in figure 3. Necessary parameters are taken from reference [21]. The formula can be simplified when D approaches 2 to [22]

$$T_c \sim \left|\frac{D}{2} - 1\right|. \tag{10}$$

Although the condensation is an essential ingredient of the superfluid transition in liquid helium-4, the condensation transition cannot be directly compared to the superfluid transition. Here, we simply note that the critical temperature curves for the superfluidity obtained from figure 2 show a similar linear behaviour.

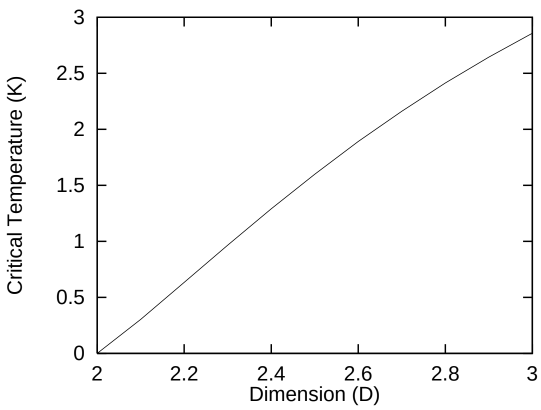

Figure 3. The critical temperatures of the ideal-Bose-gas system between $D = 2$ and $D = 3$.

In figure 1, we observe that the peaks of the specific heat curves become less and less prominent, and, eventually, the curves become flat with decreasing dimensionality. We show that this behaviour originates from a hidden hierarchy in the condensation transition with the non-integer dimensions. The first temperature derivative of the specific heat can be readily obtained from equations (7) and (8):

(a) When $T \leqslant T_c$,

$$\left(\frac{\partial}{\partial T}\right)_V \frac{C_V(T)}{N k_B} = \left(\frac{D}{2}\right)^2 \left(\frac{D}{2} + 1\right) \frac{v}{\lambda^D} \frac{g_{D/2+1}(1)}{T}. \tag{11}$$

(b) When $T > T_c$,

$$\left(\frac{\partial}{\partial T}\right)_V \frac{C_V(T)}{Nk_B} = -\left(\frac{D}{2}\right)^2 \frac{1}{T} \frac{g_{D/2}(z)}{g_{D/2-1}(z)} \times \left\{1 - \left(\frac{D}{2}+1\right)\frac{g_{D/2+1}(z)g_{D/2-1}(z)}{[g_{D/2}(z)]^2} + \frac{D}{2}\frac{g_{D/2}(z)g_{D/2-2}(z)}{[g_{D/2-1}(z)]^2}\right\}. \tag{12}$$

The first derivative of C_V is plotted in figure 4 in arbitrary units. The curves again show the shift of T_c with decreasing dimensionality. Also, it is shown that the discontinuity of $(\partial C_V(T)/\partial T)_V$ disappears with decreasing dimensionality. In order to investigate this behaviour more closely, we study the relation between the continuity of higher derivatives of C_V and the fractional dimensionality. Taking higher derivatives of C_V and considering the behaviour at T_c, we obtain the following relation:

$$\lim_{T\to T_c}\left[\left(\frac{\partial}{\partial T}\right)_V^n C_V^-(T) - \left(\frac{\partial}{\partial T}\right)_V^n C_V^+(T)\right] = \lim_{\eta\to 0}\sum_{j=1}^{n} a_{nj}\eta^{j+2-(j+1)D/2} \tag{13}$$

where the coefficients a_{nj} are finite constants, and C_V^- and C_V^+ are the specific heats below and above T_c. This formula can also be proved by the mathematical induction method (see the appendix). The hierarchy of the condensation transition with the dimensionality obtained from the above formula is summarized at table 1. This table explains the physical origin of the tendency towards roundness of the specific heat curves with decreasing dimensionality.

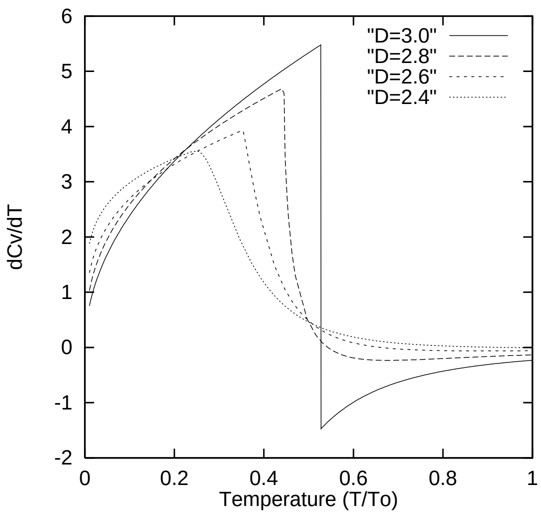

Figure 4. Plots of the first derivatives of the specific heat functions of the ideal Bose gas between $D = 2$ and $D = 3$. There are no cusps when $D < 3$. The units for the y-axis are arbitrary.

The condensate fraction in D dimensions can readily be obtained from equations (6) and (9) as

$$\frac{n_0}{n} = 1 - \left[\frac{T}{T_c(D)}\right]^{D/2}. \tag{14}$$

The curves are shown for several non-integer dimensions in figure 5. It is shown that the curves with higher dimensions have higher T_c and the curves do not cross each other. As mentioned above, the condensation fraction cannot be directly compared to the superfluid

Table 1. The hierarchy of the condensation transition between $D = 2$ and $D = 3$. The symbol 'c' stands for being continuous at T_c, and 'd' for being discontinuous at T_c. '*Class*' stands for the class of function.

Dimension	C_V	$(\partial/\partial T)C_V$	$(\partial/\partial T)^2C_V$	$(\partial/\partial T)^3C_V$	$(\partial/\partial T)^4C_V$	$\cdots$	*Class*
$D = 3$	c	d					C^0
$8/3 \leqslant D < 6/2$	c	c	d				C^1
$10/4 \leqslant D < 8/3$	c	c	c	d			C^2
$\vdots$							$\vdots$
$2(j+2)/(j+1) \leqslant D < 2(j+1)/j$	c	c	c	c	$\cdots(d)$		C^{j-1}
$\vdots$							$\vdots$
$D = 2$	c	c	c	c	c	$\cdots$	C^∞

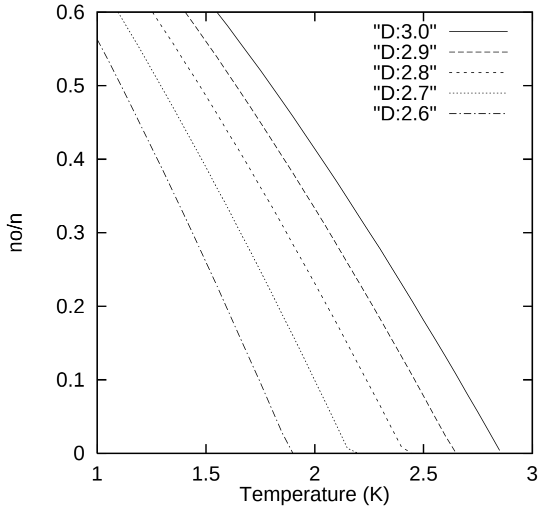

Figure 5. The condensate fraction in D-dimensional space. $D = 2.6, 2.7, 2.8, 2.9, 3.0$ from left to right.

fraction. However, here, we again note that the superfluid fraction of liquid helium-4 shows the same qualitative behaviour with the condensation fraction as shown in figure 5.

4. Discussion

Physical properties of the ideal Bose gas in non-integer dimensions between $D = 2$ and $D = 3$ are studied theoretically. Detailed calculation of the specific heat shows that there is a hidden hierarchy of the condensation transition with changing fractional dimensionality. The results obtained on the specific heat and the condensed density of the ideal Bose gas in non-integer dimensions are shown to have similar characteristics to those of liquid helium-4 in porous media. This similarity may not be totally unexpected. Careful examination of the experimental results on liquid helium-4 in porous media reveals that the salient features are almost independent of the materials and mutual interactions, thus suggesting that the dominant contributions may arise from geometrical factors.

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Appendix

Here, we prove equation (13) using the mathematical induction method. First, we give some useful relations for the Bose-gas function which are needed for the derivation:

$$g_s(z) \sim \begin{cases} z & z \to 0^+ \\ \Gamma(1-s)(-\ln z)^{s-1} + \zeta(s) & z \to 1^- \end{cases} \tag{A.1}$$

where ζ is the Riemann-zeta function; and

$$\left(\frac{\partial z}{\partial T}\right)_V = -\frac{D}{2}\frac{z}{T}\frac{g_{D/2}(z)}{g_{D/2-1}(z)}. \tag{A.2}$$

We can use the above relations for $(\partial\eta/\partial T)_V$, too. We put $z = \mathrm{e}^{-\eta}$; then $z \to 1$ as $\eta \to 0$.

We introduce a differential operator defined by

$$\Delta^n(T) \equiv \left(\frac{\partial}{\partial T}\right)_V^n C_V^-(T) - \left(\frac{\partial}{\partial T}\right)_V^n C_V^+(T) \tag{A.3}$$

where C_V^- and C_V^+ are the specific heats below and above T_c. For convenience, we drop the limit notation of '$\lim_{\eta\to 0}$ (or $T \to T_c$)' during the proof.

(i) When $n = 1$,

$$\Delta^1(T_c) = a_{11}\eta^{3-D}. \tag{A.4}$$

This is clearly true from equation (13) and the known result for $D = 3$.

(ii) We assume that equation (13) is true for any positive integer k. Then

$$\Delta^k(T_c) = \sum_{i=1}^{k} a_{ki}\eta^{i+2-[(i+1)/2]D}. \tag{A.5}$$

Using equations (A.1) and (A.2), we obtain

$$\begin{aligned} \Delta^{k+1}(T_c) &= \sum_{i=1}^{k} a_{ki}\left(i+2-\frac{i+1}{2}D\right)\eta^{i+1-[(i+1)/2]D}\left(\frac{\partial\eta}{\partial T}\right)_V \\ &= \sum_{i=1}^{k} a_{ki}\left(i+2-\frac{i+1}{2}D\right)\frac{D\zeta(D/2)}{2T_c\Gamma(2-D/2)}\eta^{i+3-[(i+2)/2]D} \\ &= \sum_{j=2}^{k+1} a_{k+1,j-1}\left(j+1-\frac{j}{2}D\right)\frac{D\zeta(D/2)}{2T_c\Gamma(2-D/2)}\eta^{j+2-[(j+1)/2]D} \\ &= \sum_{j=2}^{k+1} a_{k+1,j}\eta^{j+2-[(j+1)/2]D}. \end{aligned} \tag{A.6}$$

a_{ki} satisfies the recurrence relation

$$a_{k+1,j} = \frac{(j+1-(j/2)D)D\zeta(D/2)}{2T_c\Gamma(2-D/2)}a_{k+1,j-1} \tag{A.7}$$

where $j = 2, 3, 4, \ldots, k+1$.

Therefore, (i) and (ii) enable us, for any positive integer n, to write

$$\Delta^n(T_c) = \lim_{\eta\to 0} \sum_{j=1}^{n} a_{nj}\eta^{j+2-[(j+1)/2]D} \tag{A.8}$$

which completes the proof.

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